Historia Mathematica 8 (1981) 161-188

THE HISTORY OF DIFFERENTIAL FORMS FROM CLAIRAUT TO POINCARÉ

BY VICTOR J. KATZ
UNIVERSITY OF THE DISTRICT OF COLUMBIA
WASHINGTON, D.C.

SUMMARIES

The history of differential forms is examined from its origins in the work of Clairaut on the theory of differential equations through the end of the 19th century. In particular, we explore the developments leading to the concept of the exterior derivative, the Poincaré lemma and its converse, and the notion of the period of an exact differential. We also note that whereas the original motivation for the idea of the integral of an n-form lay in the theory of complex variables, much of its development was done by physicists.

Wir untersuchen die Geschichte der Differentialformen von den Anfängen in der Arbeit Clairauts über die Theorie der Differentialgleichungen bis zum Ende des neunzehnten Jahrhunderts. Insbesondere betrachten wir die Entwicklungen, die zu dem Begriff der äusseren Ableitung, dem Poincaréschen Lemma und seiner Umkehrung und dem Begriff der Periode eines exakten Differentials führten. Auch bemerken wir, dass, obgleich die ursprüngliche Motivierung der Idee des Integrals einer n-Form in der Theorie der komplexen Veränderlichen lag, die Physiker viel zur Entwicklung beitrugen.

Cet article traite de l'histoire des formes différentielles depuis ses origines, dans les travaux de Clairaut sur la théorie des équations différentielles, jusqu'à la fin du dix-neuvième siècle. Plus précisément, nous nous concentrons sur la succession d'évènements mathématiques aboutissant au concept de dérivée extérieure, au lemme de Poincaré et à sa réciproque et à la notion de période d'une différentielle exacte. Nous faisons aussi ressortir que quoique la motivation première de l'idée d'integrale d'une n-forme vienne de la théorie des fonctions de variables complexes, la majeure partie de son développement fut l'oeuvre de psysiciens.

0315-0860/81/020161-28$02.00/0

INTRODUCTION

Differential forms are defined (loosely) by Flanders [1963] as "things which occur under integral signs," i.e., expressions of the form

$$\omega = \Sigma f_{\alpha_1\alpha_2\ldots\alpha_k} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_k},$$

where the summation is taken over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$, and the f's are functions in n-space. It is the purpose of this article to discuss the origins of this concept and its development up to the end of the 19th century. We will trace the development of the idea of the exterior derivative, the Poincaré lemma and its converse, and the notion of the period of an exact differential, seeing that the motivation for the concepts of, first, integrals of one-forms and, later, integrals of n-forms lay in the theory of complex variables. Further, we will see that throughout much of the 19th century the subjects of line and surface integrals were not part of pure mathematics at all, but lay in the domain of physics.

After considering one-forms and line integrals and, then, two-forms and surface integrals, we will look at generalizations of the ideas developed in the two special cases.

ONE-FORMS

The simplest differential form and the first to be considered (in the mid-18th century) is the one-form in two variables, i.e., $Adx + Bdy$, where A and B are functions in two-space. For the 18th-century mathematicians, the equation $Adx + Bdy = 0$ was simply another way of writing the differential equation $dy/dx = -A/B$. They were thus interested in the conditions under which a function $f(x,y)$ exists such that $\partial f/\partial x = A$ and $\partial f/\partial y = B$. If such a function exists, then $f(x,y) = c$ is a solution to the equation $Adx + Bdy = 0$.

The first mathematician to consider this form in detail was Alexis Claude Clairaut. In two papers [1739, 1740] he proved that the necessary and sufficient condition that $Adx + Bdy$ be the differential of a function is that $dA/dy = dB/dx$. (This is Clairaut's notation; our current partial derivative notation dates from the 1840s.) Clairaut noted that the idea had occurred at about the same time to Alexis Fontaine (who never seems to have published it) and to Leonhard Euler.

Clairaut proved the necessity of the condition by an explicit calculation. To Clairaut a function of two variables was a (possibly infinite) series of terms $ax^my^n + bx^py^q + cx^ry^s + \cdots$. Hence he simply calculated the differential of ax^my^n to be $max^{m-1}y^ndx + nay^{n-1}x^mdy$ and then noted that the derivative of $max^{m-1}y^n$ with respect to y equals that of $nay^{n-1}x^m$ with respect to x. The result follows by a form of induction. More interesting for its later use, however, is his sufficiency proof.

Assuming that $dA/dy = dB/dx$, Clairaut asserted that the desired function was $\int Adx + p(y)$, where by the first term he meant any function whose derivative with respect to x is A, and by the second, some function of y alone. To show that this was correct he took its differential: $Adx + dy\int(dA/dy)dx + dp$. Since $dA/dy = dB/dx$ and $\int(dB/dx)dx$ is $B + q(y)$, the differential becomes $Adx + Bdy + dp + qdy$. So if $p(y)$ is chosen to be $-\int q(y)dy$, the expression becomes $Adx + Bdy$, as desired. In other words, Clairaut reduced the original two-variable problem to an ordinary one-variable differential equation, which he assumed to be solvable. One may note further that since $q(y) = \int(dB/dx)dx - B$, the desired function can be written as $\int Adx + \int Bdy - \int dy[\int(dB/dx)dx]$. In fact, Cauchy in [1823] replaced the indefinite integrals by definite integrals taken from a fixed point (x_0, y_0) to a varying endpoint (x,y) and wrote the solution as

$$f(x,y) = \int_{x_0}^{x} A(x,y)dx + \int_{y_0}^{y} B(x_0,y)dy. \tag{1}$$

The use of the definite integral became the standard textbook method for this proof.

Clairaut, in his paper of 1740, extended his result to one-forms in three variables. Just as in the two-variable case, a solution of $Mdx + Ndy + Pdz = 0$ was to be a function f of three variables whose differential was the given form. Clairaut showed that this was possible if and only if $dM/dy = dN/dx$, $dM/dz = dP/dx$, and $dN/dz = dP/dy$. His proof was similar to the one given for the two-variable case. Indeed, the necessity proof required only a reduction to that case, while the sufficiency proof started with $f = \int Mdx$ and used the results and methods of the two-variable case to show that the differential of $\int Mdx$ differed from $Mdx + Ndy + Pdz$ only by a complete differential in y and z.

Finally, Clairaut stated the result for any number of variables: $Mdx + Ndy + Pdz + Qdu + Rds + \cdots$ is integrable if and only if any two terms form a complete differential in those two variables; i.e., $dM/dy = dN/dx$, $dM/dz = dP/dx$, and so forth, for every combination of letters and functions.

For future reference we may note that, in modern notation, Clairaut had proved that if ω is a one-form in n variables, then $d\omega = 0$ if and only if $\omega = df$, where f is a function. (Strictly speaking, of course, this result is not always true for single-valued functions. In fact, as Jean d'Alembert [1768] observed, the example $(xdy - ydx)/(x^2 + y^2)$ shows that if the coefficients of the differential form are not continuous everywhere, the integral may not be single-valued. It took nearly another century, however, for this idea to be exploited.)

When 18th-century mathematicians considered an integral of the form $\int Adx + Bdy$, they meant simply a function whose differential was $Adx + Bdy$, assuming such a function existed; whereas today, the expression designates a line integral. Although it was not until the early 1850s that $\int Adx + Bdy$ took on this modern meaning, the ideas of the line integral and the integral over a curve were being developed long before that time. The latter concept, an integral of the type $\int fds$, first appeared in the 18th century. As early as [1760], Lagrange noted that $ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$ was an element of a curve in three-space. The curve in question represented a wire, and Lagrange was trying to determine its movement if one end is fixed and the rest is subject to certain forces. He was therefore led to consider certain integrals with respect to ds. Laplace [1799, 69] considered similar integrals of forces acting on bodies along curves.

In his *Mécanique analytique* Lagrange [1811] came somewhat closer to our modern usage when he considered the rectangular components X, Y, and Z of the forces acting on points of a wire. He noted that the element of wire dm is proportional to the element ds of the curve and then considered what he called the sum of the moments of all the forces relative to the total length of the wire, namely, the integral $\int (X\delta x + Y\delta y + Z\delta z)dm$, where δ is a differential "representing only the infinitely small spaces which each point may traverse in supposing that the situation of the body varies infinitesimally little" [Lagrange 1811, 84].

With a modern interpretation, this expression can be made into a line integral, but Lagrange himself did not do this. The use of line integrals in physics became common only forty years later. Thus, despite the work of Lagrange, the chief motivation for the development of this notion was its use in complex integrals.

In 1811 Gauss wrote a letter to Bessel [Gauss 1811] in which he discussed the integration of complex functions over curves in the complex plane; but he never published anything substantial in this area. It remained for Cauchy to develop this idea. In his paper [1825], Cauchy carefully defined $\int_{a+ib}^{c+id} f(z)dz$ as a limit. To calculate this integral, he had to describe, analytically, a path connecting the two points in the complex plane. He did so by letting $x = \phi(t)$ and $y = \psi(t)$, $\alpha \leq t \leq \beta$, showing that if $z = x + iy$, the desired integral, $\int_\alpha^\beta f(z)(dz/dt)dt$ may be written in the form

$$\int_{\alpha}^{\beta} [\phi'(t) + i\psi'(t)]f(\phi(t) + i\psi(t))dt.$$

If we write $f(z) = f(x + iy) = u(x + iy) + iv(x + iy)$ and $dz = dx + idy$, an easy calculation shows that this integral is identical to the (complex) line integral $\int udx - vdy + i\int udy + vdx$. Although Cauchy did not carry out this calculation, he did demonstrate, by a variational idea, the "Cauchy integral theorem," namely, that if $f(z)$ is bounded and analytic in a region, then the integral $\int_{a+ib}^{c+id} f(z)dz$ has the same value no matter what (differentiable) path is taken from $a + ib$ to $c + id$.

During the next twenty years, Cauchy published at least one paper [Cauchy, 1831] in which he performed the equivalent of a line integral around a circle in the plane, although the concept was not clearly articulated in either mathematical or physical works of the period [1]. It was not until 1846, however, that Cauchy wrote explicitly about integrals over curves, and then the curves lay in n-space. In a note in *Comptes Rendus* [Cauchy, 1846a], in fact, the curves, Γ, over which the integrals are to be performed are boundaries of surfaces, S, lying in a space of any (finite) dimension. One of his results states that if $Adx + Bdy + Cdz + \cdots$ is an exact differential [2], then $\int[A(dx/ds) + B(dy/ds) + \cdots]ds$ does not change when Γ varies "by insensible degrees," so long as the functions A, B, C, ... remain finite and continuous in the entire region in which the curve varies.

The second major theorem of Cauchy [1846a] asserts that if the function $k = A(dx/ds) + B(dy/ds) + \cdots$ fails to be finite and continuous solely at the points P', P'', ... in S, and if α, β, ... are closed curves surrounding these points, respectively, then $\int_{\Gamma} kds = \int_{\alpha} kds + \int_{\beta} kds + \cdots$. In particular, if there are no such points inside S, then $\int_{\Gamma} kds = 0$.

Cauchy did not provide proofs of these results, stating only that they could be based on formula (1) and its analog with the variables interchanged. Presumably he meant that if there were no singularities of A and B within the rectangle whose opposite corners are (x_0,y_0) and (x,y), then the function $f(x,y)$ given by (1) is the same as the function

$$g(x,y) = \int_{y_0}^{y} B(x,y)dy + \int_{x_0}^{x} A(x,y_0)dx.$$

Since both f and g represent line integrals along opposite pairs of sides of the rectangle, the last statement of the previous paragraph is true for integrals around rectangles. The proofs of the remaining statements, even for more variables, are not difficult to carry out, although it is not clear how Cauchy meant to make the transition from rectangles to arbitrary curves.

Cauchy observed further that in the two-variable case

$$\int_{\Gamma} kds = \iint_{S} [(dA/dy) - (dB/dx)]dxdy, \tag{2}$$

and therefore in the case of an exact differential the right side vanishes, hence also the left side. This result implies "Cauchy's integral theorem," of which we have spoken above.

It is important to note that Cauchy never wrote, as we do today, an expression of the form $\int Adx + Bdy$, where the domain of integration is a curve. For Cauchy such an expression could only have meant the sum of integrals along the x axis and y axis, respectively. To integrate over a curve, Cauchy always used the differential element ds. Hence line integrals in our sense, though implicit in his work, were not explicitly defined by Cauchy.

In [1846b] Cauchy first defined what we now call *periods of integrals*. Observing that if $\int kds$ is taken around a curve which encloses an isolated singular point ($k = A(dx/ds) + B(dy/ds)$, with $Adx + Bdy$ exact), the integral is increased by a fixed amount I on each revolution, he called this value an *index of periodicity*. If there are several isolated singular points, with corresponding "periods" I, I', I'', ..., then $\int kds$ can be written as $\pm mI \pm m'I' \pm m''I'' \pm \cdots$, where the m's are positive integers indicating the number of times the path of integration goes around the corresponding singular point.

It remained, however, for Bernhard Riemann to clarify and to prove Cauchy's results, at least in the two-dimensional case. As part of this process, he introduced the basic ideas of what we now call the topology of a Riemann surface. In other words, instead of concentrating on the points of discontinuity of the coefficient functions of the differential form, he focused his attention on the connectedness of the domains over which they were defined. In 1851, he simply sketched this idea in his dissertation, explaining it more fully in [Riemann 1857].

Before discussing the contents of [Riemann 1857], it should be mentioned that in his dissertation, Riemann, like Cauchy before him, wrote only integrals of functions over curves, e.g., $\int(X \cos \xi + Y \cos \eta)ds$, and not explicit integrals of one-forms [Riemann 1851]. In [Riemann 1857], however, the latter integrals do occur. It was the use of line integrals in physics which seems to have inspired this change. (See, however, the section on Ostrogradskii below.) Clerk Maxwell [1855, 191] had noted that if α, β, and γ are the rectangular components of the "intensity of electric action," ε, and if l, m, and n are the corresponding direction cosines of the tangent to the curve, then $\varepsilon = l\alpha + m\beta + n\gamma$, and hence $\int \varepsilon d\sigma$ can be written as $\int \alpha dx + \beta dy + \gamma dz$. A year later Charles Delaunay, in his *Treatise on Rational Mechanics*, discussed the work done by a force acting along a curve; namely, if F is a force and F_1 its tangential component, then the work done along the curve is $\int F_1 ds$; on the other hand, if X, Y, and Z are the components of F parallel to the coordinate axes, then the latter integral can be written as $\int Xdx + Ydy + Zdz$ [Delaunay 1856, 167-171]. Very quickly thereafter this became standard notation in physics.

Riemann was not concerned with physics in his publication of 1857, nor did he comment on his own change of notation. But his mathematical ideas were extremely significant. Here the notion of *multiple connectedness* was first introduced, and its relation to the integration of one-forms was made explicit. Riemann began by observing that the integral of an exact differential $Xdx + Ydy$ vanished when taken over the perimeter of a region [3] of the [Riemann] surface R which covers the x-y plane. (This follows from (2).) He continued:

> *Hence the integral $\int Xdx + Ydy$ has the same value when taken between two fixed points along two different paths, provided the two paths together form the entire boundary of a region of R. Thus, if every closed curve in the interior of R bounds a region of R, then the integral always has the same value when taken from a fixed initial point to one and the same endpoint, and is a continuous function of the position of the endpoint which is independent of the path of integration. This gives rise to a distinction among surfaces: simply connected ones, in which every closed curve bounds a region of the surface ... and multiply connected ones for which this does not happen.* ([Riemann 1857]; translation in Birkhoff 1973, 52-53)

Following this, Riemann defined multiple connectedness: "A surface R is said to be $(n + 1)$-ply connected when n closed curves $A_1, A_2, \ldots, A_n$ can be drawn on it which neither individually nor in combination bound a region of R, while if augmented by any other closed curve A_{n+1}, the set bounds some region of R" [Birkhoff 1973, 53]. He also noted that such a surface R is changed into an n-ply connected surface by any cut (a line going from one boundary point, through the interior, to another boundary point) which does not disconnect it, hence into a simply connected surface R' by n successive cuts which do not disconnect it.

Using the idea of cuts, Riemann described exactly what happens when an exact differential is integrated in an $(n + 1)$-ply connected region R. Since the associated region R' is simply connected, $Z(x,y) = \int Xdx + Ydy$ is a continuous function in R' (where the integration is performed along any curve, starting from a given point, which remains in R'). However, whenever the path of integration crosses a cut, the value changes by a fixed number depending upon the cut; in fact, there are n independent numbers, one for each cut. These numbers are, of course, a generalization of Cauchy's indices of periodicity.

The notions of multiple connectedness and of line integrals were immediately exploited by the physicists. In fact, for the next thirty years line integrals appeared mainly in the domain of physics. Although used occasionally in mathematics research papers (as we shall see), line integrals cannot be found in any mathematics texts until Hermann Laurent's *Traite d'analyse* [1888]. On the other hand, two of the most important physical texts of this period, Thomson and Tait's *Treatise on Natural Philosophy* [1867] and Maxwell's *Treatise on Electricity and Magnetism* [1873], both contain discussions of this important idea.

The concept of multiple connectedness also took on important physical meaning. Hermann Helmholtz extended Riemann's definition to three dimensional regions:

> *An n-ply connected space is one which can be cut through by n-1, but no more, surfaces without being separated into two detached portions.* [Helmholtz 1858, 27].

In other words, Helmholtz' surfaces replaced Riemann's cuts. Helmholtz observed that certain important theorems in potential theory--both in fluid dynamics and in electromagnetism--failed to hold in a multiply connected region precisely because integrals of exact differentials could not then be considered as single-valued functions. Helmholtz' important paper was translated into English in 1868, and immediately thereafter his ideas were extended by Thomson and Maxwell.

William Thomson [1869] fully explained what happened when line integrals were taken in an n-ply connected three-dimensional space, correctly stating the theorems mentioned above for such spaces. Thomson was mainly interested in fluid dynamics, and his aim was to investigate the motions of "a finite mass of incompressible frictionless fluid completely enclosed in a rigid fixed boundary.... The containing vessel may be either simply or multiply [connected]" [Thomson 1868, 13]. Thomson illustrated the discussion with pictures of pretzel-like regions and interconnected rings. He used Helmholtz' definition of an n-ply connected space to define numbers which are similar to Riemann's constants and Cauchy's indices: Let $Fds = udx + vdy + wdz$ be an exact differential and β_j be one of Helmholtz' barrier surfaces. Then if points P and Q are "each infinitely near a point B of β_j, but on the two sides of this surface," κ_j is defined to be $\int Fds$ taken along any curve in the space joining P and Q without cutting any barrier β. Thomson noted that this value is the same for any such curve and for any point B on β_j [Thomson 1869, 43-44].

Clerk Maxwell [1873] further generalized this idea. He noted that the line integral around a closed curve which passes only through the jth barrier in a given direction was κ_j. Similarly, if a closed curve Γ passes through the jth barrier m_j times, then the corresponding line integral will be

$$\int_\Gamma Fds = m_1\kappa_1 + m_2\kappa_2 + \cdots + m_n\kappa_n. \tag{3}$$

Meanwhile, Enrico Betti [1871] had proved a result more general than (3). In fact, Betti discussed the connectivity of spaces of n dimensions. Before considering his work we must look first at the notions of *two-forms* and *surface integrals*.

TWO-FORMS

A two-form in three-space is an object of the form $Adydz + Bdzdx + Cdxdy$, where A, B, and C are functions of the three variables x, y, and z. They occur under the integral sign in what today are called *surface integrals*.

Surface integrals first appeared early in the 19th century in the context of converting volume integrals to integrals over surfaces. Since the "volume element" is $dxdydz$ and surfaces are two dimensional, it seemed natural to express the integrals over surfaces in terms of products of pairs of the differentials dx, dy, and dz. Thus it became necessary to relate such products to the surface element dS. The motivation for this study evidently came from physics.

As in the case of line integrals, we must distinguish between the two related ideas of *integrals over surfaces* and *surface integrals*. The first notion was already in use, to some extent, in the late 18th century. In the first edition of his *Mécanique analytique* Lagrange [1788] mentioned the surface element and was able to write it explicitly in the case of a surface given by $z = f(x,y)$; here $dS = dxdy(1 + p^2 + q^2)^{\frac{1}{2}}$, where $p = \partial f/\partial x$ and $q = \partial f/\partial y$. But it was not until the second edition [Lagrange 1811] that he introduced the notion of a general surface integral. Here Lagrange noted that if the tangent plane at the surface element dS makes an angle γ with the x-y plane, then by simple trigonometry $dxdy = \cos\gamma\ dS$. Hence an integral of the form

$\int A dx dy$ is equal to one of the form $\int A \cos \gamma \, dS$, the first being taken over a region of the plane, the second over the corresponding region of the surface. Similarly he noted that if β is the angle the tangent plane makes with the x-z plane and α the angle it makes with the y-z plane, then $dxdz = \cos \beta \, dS$ and $dydz = \cos \alpha \, dS$. Lagrange observed further that the angles α, β, and γ are the same as the angles which a line perpendicular to the surface elements makes with the x, y, and z axes, respectively. Lagrange had been studying the laws of equilibrium of fluids surrounding solids; his integrals with respect to dS represent the sum of moments of certain forces applied to points of the surface of the fluid. He used the transformations described above to rewrite such integrals as true surface integrals, although he did not make precise the exact domain over which the surface integral should be calculated.

Gauss [1813], on the other hand, was interested in the gravitational attraction of an elliptical spheroid, although mathematically he did much the same as Lagrange. However, Gauss was careful to note that, for example, $dxdz = \pm\cos \beta \, dS$, where the sign is positive if β is acute, negative if β is obtuse. In other words, since $dxdz$ is an element of area, it is always positive and could, of course, be also written as $dzdx$. It was not until much later that any real meaning was attributed to the change in order. In any event, Gauss wrote his integrals in the form

$$\int (A \cos \alpha + B \cos \beta + C \cos \gamma) dS \qquad (4)$$

and used such integrals to express volumes in theorems which are special cases of what is now called the divergence theorem [Katz 1979].

Moreover, Gauss went further than Lagrange in showing how to calculate an integral with respect to dS. Namely, for a surface given parametrically by $x = x(p,q)$, $y = y(p,q)$, and $z = z(p,q)$, he showed, using a geometrical argument, that the area element dS is equal to

$$dpdq \left[\frac{\partial(y,z)}{\partial(p,q)}^2 + \frac{\partial(z,x)}{\partial(p,q)}^2 + \frac{\partial(x,y)}{\partial(p,q)}^2 \right]^{\frac{1}{2}}$$

and hence any integral with respect to dS can be reduced to an integral of the form $\int f dpdq$, where f is "either explicitly or implicitly a function of the two variables p, q" [Gauss 1813, 15].

By the 1820s integrals of form (4) began to appear in the works of other mathematicians. In particular, Ostrogradskii [1826], (refered to in [Yushkevich 1965; Stolze 1978]), Green [1828], and Poisson [1829] all used such integrals in their statements of the divergence theorem and related theorems. But again (as in the analogous case of line integrals and integrals over curves), explicit integrals of two forms did not appear, only the corresponding integrals of functions over surfaces. An integral of the form $\int f dS$ would be related to one of the form $\int g dy dz$, but the latter was always understood to be taken in the y-z plane. It was not until the 1850s that English physicists began to write, explicitly, integrals of the type $\int A dy dz + B dz dx + C dx dy$, where the domain of integration was the surface itself. Even the important surface integral,

$$\iint (\partial Z/\partial y - \partial Y/\partial z) dy dz + (\partial X/\partial z - \partial Z/\partial x) dz dx + (\partial Y/\partial x - \partial X/\partial y) dx dy, \qquad (5)$$

which recurred time and again in physical works of the period, first appeared as an integral with respect to dS; namely,

$$\iint [l(\partial Z/\partial y - \partial Y/\partial z) + m(\partial X/\partial z - \partial Z/\partial x) + n(\partial Y/\partial x - \partial X/\partial y)] dS, \qquad (6)$$

where l, m, and n, are the appropriate direction cosines. The equality of (6) (taken over a surface) with

$$\int (X\, dx/ds + Y\, dy/ds + Z\, dz/ds) ds$$

(taken over the boundary curve of that surface), i.e., Stokes' theorem, first appeared in 1854 and was used frequently thereafter [Katz, 1979].

The expressions in the integrand of (6) are, of course, the very expressions whose vanishing implies that a one-form $Xdx + Ydy + Zdz$ is an exact differenital. Stokes and Thomson, among others, studied one-forms for which the expressions do not vanish. Physically, such expressions represent the components of the rotation of, for example, a fluid whose velocity vector is (X,Y,Z). This fact was apparently first noticed by Cauchy [1843] and Stokes [1845] and was later developed in greater detail. In particular, Stokes [1849] considered these expressions in his work on diffraction.

Since the integrand in (5) is (using modern terminology) the exterior derivative of the one-form, $Xdx + Ydy + Zdz$, we will first consider the mathematical aspects of its use. After Jacobi [1836] observed that

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) = 0$$

(an obvious result of the rules for partial differentiation), George Stokes [1849] proved what amounts to the converse, namely, that if A, B, and C are functions satisfying

$$\partial A/\partial x + \partial B/\partial y + \partial C/\partial z = 0,$$

then there exist functions X, Y, and Z such that

$$A = \partial Z/\partial y - \partial Y/\partial z \qquad B = \partial X/\partial z - \partial Z/\partial x \qquad C = \partial Y/\partial x - \partial X/\partial y.$$

Stokes' proof required the solution of Laplace' equation, $\nabla U = f$. Thomson [1851] sketched a simpler proof, the details of which we now present. Let

$$X = \int (B/3)dz - \int (C/3)dy, \qquad Y = \int (C/3)dx - \int (A/3)dz,$$

$$Z = \int (A/3)dy - \int (B/3)dx,$$

where the integrations are all partial with respect to the given variable, so that in each case the "constant of integration" is a function of the remaining variables. To be explicit, let us rewrite these as

$$X = \int (B/3)dz + f_1(x,y) - \int (C/3)dy + f_2(x,z),$$

$$Y = \int (C/3)dx + g_1(y,z) - \int (A/3)dz + g_2(x,y),$$

$$Z = \int (A/3)dy + h_1(x,z) - \int (B/3)dx + h_2(y,z).$$

Then,

$$\begin{aligned}\partial Z/\partial y - \partial Y/\partial z &= A/3 - (1/3)\int (\partial B/\partial y)dx + \partial h_2/\partial y \\ &\qquad - (1/3)\int (\partial C/\partial z)dx - \partial g_1/\partial z + A/3 \\ &= 2A/3 + (1/3)\int (\partial A/\partial x)dx + \partial h_2/\partial y - \partial g_1/\partial z \\ &= A + k(y,z) + \partial h_2/\partial y - \partial g_1/\partial z.\end{aligned}$$

It is then simple to choose g_1 and h_2 such that $\partial g_1/\partial z - \partial h_2/\partial y = k$, thus giving the desired result. (Note that g_1 and h_2 are functions of y and z only.) A similar calculation works in the two other cases [4].

In modern terminology Jacobi's result and that of Stokes and Thomson together prove that if ω is a two-form in three variables, then $d\omega = 0$ if and only if $\omega = d\eta$, where η is a one-form. In Clairaut's earlier [1740] proof of a simpler case of this result, an important idea was the reduction of the problem from two variables to one; in Thomson's proof, the reduction from a problem in three variables to a problem in two variables also played a central role. Later this result and proof were further generalized.

Thomson needed this result in a discussion of *solenoidal* and *lamellar* distributions of magnetism; the first is a distribution (A, B, C) where $\partial A/\partial x + \partial B/\partial y + \partial C/\partial z = 0$, and the second is one where $Adx + Bdy + Cdz$ is the differential of some function ϕ. In each case he wanted a formula for the magnetic potential V at a point. In the second case, he knew how to find V in terms of ϕ, but in the first it turned out that he needed to find the functions X, Y, and Z in order to obtain a usable expression for V. Mathematically, a "lamellar" distribution occurs when the form $\omega = Adx + Bdy + Cdz$ satisfies $d\omega = 0$; while a "solenoidal" distribution occurs when the form $\eta = Adydz + Bdzdx + Cdxdy$ satisfies $d\eta = 0$. Nearly fifty years were to pass before the unification of these two ideas under the idea of the exterior derivative took place.

In any case, by the mid-19th century, the basic ideas of line and surface integrals were well understood and were regularly used in physical applications. In fact, in the preliminary sections of [Maxwell 1873] there is a detailed discussion of most of the ideas we have considered.

THE GENERALIZATIONS OF OSTROGRADSKII AND BETTI

The work of Mikhail Ostrogradskii on this subject is something of an anomaly. He made a number of important discoveries, but his work appears to have been totally ignored, at least in western Europe. For example, Ostrogradskii gave the first correct proof of the change-of-variable formula for double integrals [1838], found the first generalization of this formula to any number of variables, generalized the divergence theorem to n variables [1836], wrote integrals of n-forms (in our sense) [1836, 1840], and even gave a mathematical definition of a "region in n-space" [1836]. The results were reproduced later by other mathematicians, with no credit given to Ostrogradskii. This lack of recognition is difficult to understand, especially since Ostrogradskii wrote most of his papers in French, and the important work of 1836 was published in the *Journal fur die Reine und Angewandte Mathematik*, one of the most widely read journals of the day.

In particular, Ostrogradskii's generalization [1836] of the divergence theorem, with its concomitant introduction of integrals of n-forms, appeared nowhere else until Enrico Betti repeated them thirty-five years later. Ostrogradskii's generalization of the divergence theorem states that

$$\begin{aligned}\int_V (\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z + \cdots)\,dxdydz\ldots \\ = \int_S \{(P\partial L/\partial x)/[(\partial L/\partial x)^2]^{\frac{1}{2}}\}dydz\ldots \\ + \{(Q\partial L/\partial y)/[(\partial L/\partial y)^2]^{\frac{1}{2}}dxdz\ldots \\ + \{(R\partial L/\partial z)/[(\partial L/\partial z)^2]^{\frac{1}{2}}dxdy\ldots + \cdots,\end{aligned} \tag{7}$$

where L is a function of the variables x, y, z, $\ldots$; V is the set of points $(x,y,z,\ldots)$ with $L(x,y,z,\ldots) < 0$; and S is the set of points with $L(x,y,z,\ldots) = 0$. (In modern terminology: if there are n coordinates, S is an $(n - 1)$-dimensional hypersurface bounding the n-dimensional volume V.) The integrand on the right in (7) is the first appearance of what is called today an $(n - 1)$-dimensional differential form. The expressions $(\partial L/\partial x)/[(\partial L/\partial x)^2]^{\frac{1}{2}}$, etc., are simply Ostrogradskii's way of designating the sign of the integrand. (The proof of (7) is discussed in [Katz 1979].)

In his second statement of the theorem, Ostrogradskii once more generalized earlier work by expressing an element of "hypersurface" dS by the equations

$$\frac{dS}{[(\partial L/\partial x)^2 + (\partial L/\partial y)^2 + \cdots]^{\frac{1}{2}}} = \frac{dydz\ldots}{[(\partial L/\partial x)^2]^{\frac{1}{2}}} = \frac{dxdz\ldots}{[(\partial L/\partial y)^2]^{\frac{1}{2}}}$$

$$= \frac{dxdy\ldots}{[(\partial L/\partial z)^2]^{\frac{1}{2}}} = \cdots.$$

Then the right side of (7) becomes

$$\begin{aligned}\int_S (P\partial L/\partial x + Q\partial L/\partial y + R\partial L/\partial z + \cdots) \\ \times [(\partial L/\partial x)^2 + (\partial L/\partial y)^2 + (\partial L/\partial z)^2 + \cdots]^{-\frac{1}{2}}ds\end{aligned} \tag{8}$$

Since

$$[(\partial L/\partial x)^2 + (\partial L/\partial y)^2 + \cdots]^{-\frac{1}{2}}(\partial L/\partial x,\ \partial L/\partial y,\ \ldots)$$

is a unit normal $\vec{n}$ to S, each component of this vector may be considered to be the cosine of the angle which $\vec{n}$ makes with the corresponding coordinate axis. It is then easy to see how Eq. (7), with the right side replaced by expression (8), is a direct generalization of the ordinary divergence theorem.

In 1871, Betti repeated the work of Ostrogradskii, generalizing it to multiply connected spaces. Moreover, he did the same for Stokes' theorem. Betti was strongly influenced by the work of Riemann; in 1859 he had translated Riemann's inaugural dissertation into Italian and over the next several years had done other work in the theory of complex variables. Furthermore, in letters written in 1863 to his colleague Tardy, Betti described his discussions with Riemann on multiple connectivity and stated that he had "formed an accurate idea of the matter." In these letters he also discussed several examples of these ideas [Weil 1979, quoting Loria 1915].

Betti [1871] was the first to publish a comprehensive definition of connectivity for an n-dimensional space R: for each dimension $m < n$, R is said to have *m-dimensional order of connectivity* $p_m + 1$, if there are p_m closed m-dimensional spaces $A_1, A_2, \ldots, A_{p_m}$ in R, which together do not form the boundary of a connected $(m + 1)$-dimensional region of R, while any additional closed m-dimensional space together with some subset of the A_j's forms such a boundary [5]. (A closed space, for Betti, was one without a boundary.) So, for instance, in a space whose m-dimensional order of connectivity is 1 (i.e., a space which is simply connected in the mth dimension), any closed m-dimensional space is the boundary of an $(m + 1)$-dimensional region. For $n = 2$ and $m = 1$, Betti's definition is the same as Riemann's original definition.

Again generalizing the work of Riemann, Betti showed also that to make a space simply connected in the mth dimension, one had to remove from it p_m $(n - m)$-dimensional cross sections. For example, if $m = 1$, p_1 $(n - 1)$-dimensional sections must be removed from R to make the remainder R' simply connected in the first dimension. Betti went on to compare n-fold integrals with $(n - 1)$-fold integrals, using a method similar to that of Ostrogradskii; then generalizing the ideas of Stokes, he compared integrals of one-forms with those of two-forms.

In the first case, he considered an n-dimensional region R bounded by closed $(n - 1)$-dimensional spaces $S_1, S_2, \ldots, S_t$, given respectively by equations $F_1 = 0$, $F_2 = 0$, $\ldots$, $F_t = 0$. For simplicity, we will consider only the case $t = 1$, which is precisely the case studied by Ostrogradskii. Betti considered

n functions $X_1, X_2, \ldots, X_n$ in R and aimed to express the n-fold integral

$$\Omega_n = \int_R (\partial X_1/\partial z_1 + \partial X_2/\partial z_2 + \cdots + \partial X_n/\partial z_n)\, dz_1 dz_2 \cdots dz_n \tag{9}$$

in terms of an $(n - 1)$-fold integral. More explicitly than Ostrogradskii, he gave a parametric expression $z_i = z_i(u_1, u_2, \ldots, u_{n-1})$ $(i = 1,2,\ldots,n)$ for the hypersurface S whose equation is $F = 0$, and then, via a proof quite similar to that in the earlier work, showed (in modern notation) that

$$\Omega_n = \int \Sigma (-1)^i X_i \partial (z_1 \cdots \hat{z}_i \cdots z_n)/\partial (u_1 \cdots u_{n-1})\, du_1 du_2 \cdots du_{n-1}$$

This can, of course, be rewritten as

$$\int \Sigma (-1)^i X_i dz_1 \cdots d\hat{z}_i \cdots dz_n \tag{10}$$

The equality of (9) with (10) is nearly the same as Ostrogradskii's, equality (7). The difference in sign is due to Betti's careful choice of the order of the coordinates z_i.

Betti also rewrote Ω_n in a way reminiscent of Ostrogradskii's form (8). Defining M to be

$$[\Sigma (\partial (z_1 \cdots \hat{z}_i \cdots z_n)/\partial (u_1 \cdots u_{n-1})^2]^{\frac{1}{2}}$$

and μ to be $[\Sigma (\partial F/\partial z_i)^2]^{\frac{1}{2}}$, he noted that $dS = M du_1 du_2 \cdots du_{n-1}$ and

$$\partial F/\partial z_i = (-1)^{i+1} [\partial (z_1 \cdots \hat{z}_i \cdots z_n)/\partial (u_1 \cdots u_{n-1})] \cdot [\mu/M];$$

hence

$$\Omega_n = -\Sigma X_i (\partial F/\partial z_i)(1/\mu) dS$$

a result identical (up to sign) to that of Ostrogradskii. (Note, of course, that the expression $(\partial F/\partial z_i)(1/\mu)$ are components of the unit normal vector to S.)

But Betti carried the idea further in a particular case. Given a function V such that $X_i = \partial V/\partial z_i$ for all i, he observed that

$$\Sigma(X_i \partial F/\partial z_i)(1/\mu) = \Sigma(\partial V/\partial z_i)(\partial F/\partial z_i)(1/\mu)$$

is the normal derivative dV/dp of V. Hence Betti's theorem implies that $\int_R \Sigma(\partial^2 V/\partial z_i^2)dR = -\int_S (dV/dp)dS$. Therefore if

$$\Sigma(\partial^2 V/\partial z_i^2) = 0 \qquad (11)$$

throughout R, then $\int_S (dV/dp)dS = 0$. Hence if C is any closed space which forms the boundary of a portion of R and if condition (11) is satisfied, then $\int_C (dV/dp)dC = 0$.

Betti next applied his definition of connectivity. If R has connectivity $p + 1$ in the $(n - 1)$st dimension, there are p closed $(n - 1)$-dimensional spaces $A_1, A_2, \ldots, A_p$, such that each closed $(n - 1)$-dimensional space C contained in R forms with the A's the boundary of a region of R. Setting $\int_{A_r}(dV/dp)dA_r = M_r$, Betti concluded that $\int_C (dV/dp)dC = -\Sigma M_r$, a result similar to that of Thomson and Maxwell. As a corollary he noted the following for a space R which is simply connected in the $(n - 1)$st dimension. Since two $(n - 1)$-dimensional spaces having the same boundary Γ together form a closed space, it follows that "the integral extended to any space C, contained in R, with boundary Γ will always have the same value" [Betti 1871, 156]. Of course, Betti did not have to require that condition (11) be satisfied by the function V. To derive similar results he could have used arbitrary functions X_i, assuming only that $\Sigma(\partial X_i/\partial z_i) = 0$. (As we will see below, that is precisely what Poincaré did.)

Betti also considered the case of one-forms in an n-dimensional space R, namely, forms of the type $\Sigma X_i dz_i$. He assumed that the curve γ, represented parametrically by $z_i = z_i(u)$, bounds the region C given by $z_i = z_i(v_1, v_2)$. Then he defined

$$\Omega_1 = \int_\gamma \Sigma X_i dz_i = \int \Sigma X_i (dz_i/du)du.$$

Since $(dz_i/du)du = (\partial z_i/\partial v_1)dv_1 + (\partial z_i/\partial v_2)dv_2$, we obtain

$$\Omega_1 = \int \Sigma X_i (\partial z_i/\partial v_1)dv_1 + \int \Sigma X_i (\partial z_i/\partial v_2)dv_2.$$

By a direct calculation, he showed that this expression is equal to

$$\iint [\partial(\Sigma X_i \partial z_i/\partial v_1)/\partial v_2 - \partial(\Sigma X_i \partial z_i/\partial v_2)/\partial v_1]dv_1 dv_2, \qquad (12)$$

where the latter integral is taken over the parametric space of C. After further calculation Betti concluded that

$$\int_\gamma \Sigma X_i dz_i = \iint_C \Sigma (\partial X_i/\partial z_j - \partial X_j/\partial z_i) dz_i dz_j. \tag{13}$$

This is, of course, a direct generalization of Stokes' theorem from three- to n-dimensional space.

Next Betti assumed that the connectivity of R in the first dimension was $p + 1$. This means that there are p $(n - 1)$-dimensional cross sections $s_1, s_2, \ldots, s_p$, such that on removing these sections from R, the remainder R' will be simply connected. Furthermore, there are p closed curves $L_1, L_2, \ldots, L_p$, which, respectively, meet the sections s_i, and such that any other closed curve γ forms with these L's the boundary of a two-dimensional space C. Betti used (13) to conclude that if

$$\partial X_i/\partial z_j - \partial X_j/\partial z_i = 0 \quad \text{for all } i, j, \tag{14}$$

then $\int \Sigma X_i dz_i = 0$; the integration is over the entire boundary system of C, namely, $\gamma, L_1, L_2, \ldots, L_n$. It follows that if $M_t = \int_{L_t} \Sigma X_i dz_i$, then "the integral $\int \Sigma X_i dz_i$, taken from Z_0 to Z_1 along any curve which meets certain sections s [i.e., s_j], differs from that taken along any curve from Z_0 to Z_1 which does not meet any of the sections s by the quantities M [i.e., M_j] relative to the intersected sections s; these quantities are taken to be positive or negative depending on whether they [the curves] intersect the section [while] progressing in one or the other direction [i.e., depending on the direction of the integration]." Hence, if R is simply connected in the first dimension, "the integral taken along any curve in R from Z_0 to Z_1 always has the same value" [Betti 1871, 158]. It is easy to see that this result is the same as Maxwell's (3) except, of course, that it is valid for an arbitrary number of dimensions.

THE GENERALIZATIONS OF POINCARÉ AND VOLTERRA

Conditions (14) ensuring that the line integral $\int \Sigma X_i dz_i$ is independent of the path of integration and depends only on the endpoints (in a simply conected space) were called the *integrability conditions* by Poincaré [1887]. The conditions were so named because they imply the existence of an "integral" for $\Sigma X_i dz_i$, that is, a function f such that $df = \Sigma X_i dz_i$. Poincaré went on to consider similar conditions for surface integrals in n-dimensional space. He was very explicit about his motivations for considering such conditions; both in [Poincaré 1886] and in the retrospective analysis of his work [Poincaré 1921], he stated that his aim was to generalize the work of Cauchy on functions of one complex variable to functions of two complex variables. In particular, he wanted to generalize Cauchy's integral theorem, the idea of a period, and the notion of a residue.

In [Poincaré 1887], a function, $F(\xi,\eta) = P(\xi,\eta) + iQ(\xi,\eta)$, of two complex variables, $\xi = x + iy$ and $\eta = z + it$, was introduced. Its double integral, $\iint F(\xi,\eta)d\xi d\eta$, taken over a region in complex two-dimensional space, can be expanded formally to give

$$\iint(P + iQ)(dx + idy)(dz + idt)$$
$$= \iint(P + iQ)dxdz + (iP - Q)dxdt + (iP - Q)dydz - (P + iQ)dydt$$

taken over a surface in four-space. As in the one-variable case, the Cauchy-Riemann conditions are satisfied by P and Q:

$$\partial P/\partial x = \partial Q/\partial y, \quad \partial P/\partial y = -\partial Q/\partial x, \quad \partial P/\partial z = \partial Q/\partial t \quad \partial P/\partial t = -\partial Q/\partial z,$$

and they may be used to derive

$$\partial(iP - Q)/\partial x - \partial(P + iQ)/\partial y = 0, \quad -\partial(P + iQ)/\partial x - \partial(iP - Q)/\partial y = 0,$$
$$\partial(P + iQ)/\partial t - \partial(iP - Q)/\partial z = 0, \quad \partial(iP - Q)/\partial t + \partial(P + iQ)/\partial z = 0. \tag{15}$$

Poincaré wanted to show that conditions (15) are precisely those which ensure that the given integral vanishes over a closed surface. (This is the analog of Cauchy's integral theorem for functions of two variables.)

To do this, he returned to the real case and to surface integrals in n-space, namely, integrals of the form

$$J = \iint\Sigma(X_i,X_k)dx_i dx_k, \tag{16}$$

where each symbol (X_i,X_k) denotes a function of the n variables x_1, x_2, ..., and x_n; $(X_i,X_k) = 0$ and $(X_i,X_k) = -(X_i,X_k)$ for all values of i and k; and the summation in (16) is taken over all $n(n - 1)/2$ distinct pairs of indices. Poincaré defined this integral by parametrizing the surface, thereby converting J into an ordinary double integral in the plane. It is here that Poincaré was careful to remark that the order of integration of the parametric variables is crucial. Indeed, since

$$\iint A\,dx\,dy = \iint A[\partial(x,y)/\partial(u,v)]\,du\,dv,$$

it is clear that interchanging either u and v or x and y will change the sign of the integral. We thus see the reason for Poincaré's insistence on the skew symmetry of the functions (X_i,X_k).

Poincaré went on to derive the integrability conditions he was seeking, namely, the conditions under which the integral does not depend on the surface of integration, but only on the curve which bounds the surface. These conditions turned out to be the $n(n-1)(n-2)/6$ relations

$$\partial(X_i,X_k)/\partial x_h + \partial(X_k,X_h)/\partial x_i + \partial(X_h,X_i)/\partial x_k = 0 \qquad (17)$$

[Poincaré 1887, 452]. As we noted above, the special case $n = 3$ of (17) is equivalent to one of Betti's results [Betti 1871, 156]; but Poincaré's generalizations and proofs differed from the latter. Betti used an argument comparing an n-dimensional integral to an $(n-1)$-dimensional integral, whereas Poincaré used a calculus of variations proof, thereby considering only the two-dimensional integral itself.

Moreover, the case $n = 4$ gave Poincaré the result he wanted for studying functions of two complex variables. By letting $(X,Y) = (Z,T) = 0$, $(X,Z) = (T,Y) = P + iQ$, and $(X,T) = (Y,Z) = iP - Q$, the four conditions described by (17) become identical with those of (15).

Having obtained the result for two-dimensional integrals, Poincaré immediately generalized it to integrals of higher order: given a triple integral $\iiint\Sigma(X_\alpha,X_\beta,X_\gamma)dx_\alpha dx_\beta dx_\gamma$, where the symbols $(X_\alpha,X_\beta,X_\gamma)$ are analogous to those in the two-dimensional case, the conditions of integrability, i.e., the conditions under which the integral depends only on the two-dimensional boundary of the three-dimensional space over which the integral is taken, are

$$\partial(X_\alpha,X_\beta,X_\gamma)/\partial x_\delta - \partial(X_\beta,X_\gamma,X_\delta)/\partial x_\alpha + \partial(X_\gamma,X_\delta,X_\alpha)/\partial x_\beta - \partial(X_\delta,X_\alpha,X_\beta)/\partial x_\gamma = 0.$$

Poincaré noted that similar results would hold in any dimension, with the signs between the individual terms alternating in the odd-dimensional cases and always being positive in the even-dimensional cases [Poincaré 1887, 453].

Vito Volterra [1889c] derived the same integrability conditions, though from a slightly different point of view. Volterra had been developing the theory of *functionals*--or what he called "functions of lines." Poincaré had shown that the integral of a function of two complex variables over a surface depends only on the "lines" bounding the surface. In other words, according to Volterra [1889a, 365], Poincaré's double integrals could be considered functions of lines. In a series of papers published in 1889, Volterra developed this theory and its generalization to functions of r-dimensional surfaces, after which he noted its application to complex analysis.

As part of this work, he was led to the first general statement and proof of what is today known as Poincaré's lemma and its converse. We quote his result, written in the language of systems of partial differential equations in n-space:

> *The necessary and sufficient conditions that the system of simultaneous differential equations*
>
> $$\sum_{1}^{r+1} (-1)^t \frac{\partial P_{i_1 \cdots \hat{i}_t \cdots i_{r+1}}}{\partial x_{i_t}} = p_{i_1 \cdots i_{r+1}} \tag{18}$$
>
> *be integrable is that*
>
> $$\sum_{1}^{r+2} (-1)^s \frac{\partial p_{i_1 \cdots \hat{i}_s \cdots i_{r+2}}}{\partial x_{i_s}} = 0, \tag{19}$$
>
> *Eqs. (18) are taken for every combination of* $r + 1$ *of the indices* 1, 2, ..., n*; also the p's and the P's change sign by any transposition of indices.* [Volterra 1889, 422]

Equations (19) are the same as Poincaré's conditions; the apparent differences in signs are due to the different orders in which the indices are placed. Furthermore, Volterra's solutions to the differential equations imply Poincaré's result, namely, that certain integrals depend only on the boundaries of regions.

If conditions (19) are given, then Volterra's theorem states that there exist functions P satisfying (18). The generalized form of Stokes' theorem, which Volterra had stated in [1889b], implies that for functions related by (18),

$$\int_{S_{r+1}} \Sigma P_{i_1 \cdots i_{r+1}} \alpha_{i_1 \cdots i_{r+1}} dS_{r+1} = \int_{S_r} \Sigma P_{i_1 \cdots i_r} \beta_{i_1 \cdots i_r} dS_r,$$

where S_r is the r-dimensional boundary of the $(r + 1)$-dimensional space S_{r+1}, the α's are the direction cosines of S_{r+1}, and the β's are the direction cosines of S_r. In other words, the integral on the left (which, as we have seen in the works of Ostrogradskii and Betti, can be written as the integral of an $(r + 1)$-dimensional form) depends only on an integral over the boundary S_r.

To prove the necessity of the conditions stated in Volterra's theorem requires only a direct calculation using the rules for partial derivatives. This is, of course, a simple generalization of Jacobi's earlier result in three dimensions. The sufficiency part of the proof involves a step-by-step reduction in the number of variables, in essence a generalization of Clairaut's original proof for (in Volterra's notation) the case $r = 0$ and $n = 2$. Rather than sketch his proof in all generality, we will consider only the special case $r = 1$ and $n = 4$. This case exhibits the main features of the general proof.

Six functions p_{ij}, $1 \leq i < j \leq 4$, satisfying the four conditions

$$\partial p_{jk}/\partial x_i + \partial p_{ik}/\partial x_j - \partial p_{ij}/\partial x_k = 0, \qquad 1 \leq i < j < k \leq 4, \quad (20)$$

are given, and four functions P_i, satisfying

$$-\partial P_j/\partial x_i + \partial P_i/\partial x_j = p_{ij}, \quad 1 \leq i < j \leq 4, \quad (21)$$

must be found. Volterra first chose M_4 to be an arbitrary function. Then M_1, M_2, and M_3 may be chosen to satisfy

$$\partial M_i/\partial x_4 = p_{i4} + \partial M_4/\partial x_i, \quad i = 1, 2, 3. \quad (22)$$

Volterra showed that

$$\partial(-M_j/\partial x_i + \partial M_i/\partial x_j)/\partial x_4 = \partial p_{ij}/\partial x_4, \qquad 1 \leq i < j \leq 3,$$

and therefore

$$-\partial M_j/\partial x_i + \partial M_i/\partial x_j = p_{ij} + p'_{ij}, \quad (23)$$

where p'_{ij} is a function solely of x_1, x_2, and x_3. Another straightforward calculation using (23) and (20) yields

$$-\partial p'_{23}/\partial x_1 + \partial p'_{13}/\partial x_2 - \partial p'_{12}/\partial x_3 = 0. \tag{24}$$

If functions P'_1, P'_2, and P'_3 can be found such that

$$-\partial P'_j/\partial x_i + \partial P'_i/\partial x_j = -p'_{ij}, \quad 1 \leq i < j \leq 3, \tag{25}$$

then by setting $P_i = M_i + P'_i$, $i = 1, 2, 3$, and $P_4 = M_4$, it is easy to see that the P_i satisfy (21).

Thus, Volterra reduced the problem for $r = 1$ and $n = 4$ to $r = 1$ and $n = 3$ (i.e., to solve equations (25), given condition (24)). The latter case was precisely the one considered earlier by Stokes and Thomson. For Thomson, as for Volterra, the next reduction to the case $r = 1$ and $n = 2$ enabled him to find the solution immediately, since only a single equation of type (25) had to be solved.

In this proof, Volterra assumed that equations of type (22) can always be solved, i.e., that there are always partial antiderivatives. Locally this is always true; but as d'Alembert's example showed, there are domains to which the solution cannot be extended. Nevertheless, as noted above, this was the first proof that if $\omega = d\eta$, then $d\omega = 0$; and conversely, if $d\omega = 0$, then, at least locally, there is an η with $d\eta = \omega$.

In 1887 Poincaré, like Cauchy in 1846, had considered connectivity of the domain only in the sense that for a function to have an integral equal to zero over a closed surface, it must not have any singularities either on the surface or in the domain bounded by that surface. In his fundamental paper, "Analysis Situs," however, Poincaré [1895], like Riemann in 1857, refined the notion of the connectivity of a domain. First, he defined the notions of *homology* and *Betti number*, further clarifying them four years later in [Poincaré 1899]: a homology relation exists among p-dimensional subvarieties $\nu_1, \nu_2, \cdots, \nu_r$ of an n-dimensional variety V, written $\nu_1 + \nu_2 + \cdots + \nu_r \sim 0$, if for some integer k, the set consisting of k copies of each of the ν_i constitutes the complete boundary of a $(p + 1)$-dimensional subvariety W [Poincaré 1895, 207; 1899, 291]. "Negatives" of varieties were introduced by considering orientation. Poincaré observed that homologies can be added, subtracted, and multiplied by integers. Finally, he called varieties "linearly independent" if there is no homology among them with integer coefficients [6].

Poincaré went on to define the q-dimensional Betti number P_q of V to be one more than the maximum number of independent, closed, q-dimensional subvarieties. This is nearly the same as Betti's definition of the order of connectivity: the difference is that Betti had failed to consider the possibility that a multiple of a variety was a boundary, while the variety itself was not. (Today, we would define the q-dimensional Betti number to be the number of independent closed q-dimensional subvarieties.)

Having defined the Betti numbers, Poincaré was ready to consider integrals of the form

$$\int \Sigma X_{\alpha_1 \cdots \alpha_q} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_q}, \tag{26}$$

for which he gave the appropriate definition. He repeated the conditions of integrability first given in [Poincaré 1887, 453], i.e., the conditions under which the integral (26) vanishes over any closed variety. Then he generalized this result: if the complete boundary of an $(m + 1)$-dimensional variety W is composed of k m-dimensional varieties $\nu_1, \nu_2, \ldots, \nu_k$, then, assuming the integrability conditions are satisfied, the algebraic sum of the integral (26) over the ν_i will also be zero.

Hence, since there are $P_m - 1$ independent closed m-dimensional varieties $\nu_1, \ldots, \nu_{P_m - 1}$ such that any closed variety U is (up to homology) a linear combination of these and since, therefore, a multiple of U together with the same multiple of this linear combination forms the boundary of an $(m + 1)$-dimensional variety, Poincaré concluded that the integral (26) taken over U is simply a linear combination of the values that the integral takes over the ν_i. Poincaré called these values, which are the generalizations of similar values appearing in the works of earlier authors, the *periods of the integral*. In particular, he noted that Betti had done essentially the same thing, but only for dimensions 1 and $n - 1$.

In the 19th century there was a second stream of mathematical investigations, related to the problem of Pfaff [7] and carried out by such mathematicians as Darboux and Frobenius. This stream was to merge with the investigations discussed here, culminating, at the turn of the century, with the work of Elie Cartan formalizing the theory of differential forms [8].

ACKNOWLEDGMENTS

The writer wishes to express his gratitude to the faculty and staff of the Institute for the History and Philosophy of Science and Technology at the University of Toronto for their help during the preparation of this paper while he was on sabbatical leave from the University of the District of Columbia in 1978-1979. In particular, Tom Archibald, Charles Jones, and Ed Barbeau read and commented on an earlier version. The paper was completed with the help of a faculty research grant from UDC during the summer of 1980.

NOTES

1. Cauchy's publication of 1831 was pointed out to me in a letter from Dr. F. Smithies of Cambridge University. He is currently doing research on the origins of Cauchy's integral theorem.

2. Throughout this paper the term *exact differential* will designate, according to 19th-century terminology, differentials which satisfy the Clairaut conditions. Today they would be called *closed differentials*, the term exact differential being reserved for the differential of a function.

3. Riemann gave no explicit definition of region [*Theil*] here, but the context shows that he considered it to be a subset of the surface in which his proof that $\int Xdx + Ydy = \iint (\partial Y/\partial x - \partial X/\partial y)dR$ is valid, namely, one in which every closed curve can be continuously deformed into a point.

4. In Thomson's original paper, the factor 1/3 does not appear in the proof, thereby rendering it incorrect; but this factor does appear in the reprint. It is interesting that when Maxwell [1855] reproduced the proof he referred to Thomson [1851] and also left out the factor 1/3.

5. This definition, of course, requires a theorem to show that it is consistent. Although Betti provided such a theorem, A. Tonelli [1873] pointed out that the proof was not rigorous. To correct it required some modification of the definition. The situation was not completely clarified until the work of Poincaré [1899]. (There is a detailed treatment of this point in [Pont 1974].)

6. Poincaré gave two definitions of a variety in n-space. He first defined it as the solution set of a system of equations, $F_i(x_1,\ldots,x_n) = 0$, and inequalities, $\phi_j(x_1,\ldots,x_n) > 0$, where the F_i and the ϕ_j satisfy certain differentiability conditions. His second definition was a parametric one: a variety is the image of a set of n analytic functions $x_i = \theta_i(y_1,\ldots,y_m)$, where the domain in m-space is specified by certain inequalities $\psi_k(y_1,\ldots,y_m) > 0$.

7. The problem of Pfaff is concerned with the conditions under which the number of variables in a first-order form (i.e., a Pfaffian) can be reduced by a change of variable.

8. The author is preparing a sequel to this paper in which these investigations will be described.

REFERENCES

d'Alembert, J. 1768. Sur l'équilibre des fluides. *Opuscules Mathématiques* 5, 1-40.

Betti, E. 1871. Sopra gli spazi di un numero qualunque di dimensioni. *Annali di Matematica* (2) 4, 140-158; *Opere* 2, 273-290.

Birkhoff, G. (ed.). 1973. *A source book in classical analysis*. Cambridge, Mass: Harvard Univ. Press.

Cauchy, A.-L. 1823. *Résumé des Leçons données a l'École Polytéchnique sur le calcul infinitésimal*. Paris: l'Imprimerie Royale; *Oeuvres* (2) 4.

——— 1825. *Mémoire sur les intégrales definies prises entre des limites imaginaires*. Paris; *Oeuvres* (2) 15, 41-89; a partial translation is given in [Birkhoff 1973, 33-37].

——— 1831. Formules extrait d'un mémoire presente le 27 Novembre 1831 a l'Académie des Sciences de Turin. *Bulletin de Ferussac* 16, 119-128; *Oeuvres* (2) 2, 173-183.

——— 1843. Mémoire sur les dilatations, les condensations et les rotations produites par un changement de forme dans un système de points matériels. *Exercises d'Analyse* 2; *Oeuvres* (2) 12, 343-377.

——— 1846a. Sur les intégrales qui s'étendent a tous les points d'une courbe fermée. *Comptes Rendus* 23, 251-255; *Oeuvres* (1) 10, 70-74.

——— 1846b. Considérations nouvelles sur les intégrales définies qui s'étendent a tous les points d'une courbe fermée, et sur celles qui sont prises entre des limites imaginaires. *Comptes Rendus* 23, 689-704; *Oeuvres* (1) 10, 153-168.

Clairaut, A. 1739. Recherches générales sur le calcul intégral. *Histoire de l'Académie Royale des Science avec les Mémoires de Mathématique et de Physique*, 425-436.

——— 1740. Sur l'intégration ou la construction des equations différentielles du premier ordre. *Histoire de l'Académie Royale des Science avec les Mémoires de Mathématique et de Physique*, 293-323.

Delaunay, C. 1856. *Traite de mécanique rationnelle*. Paris: Langlois & Leclercq.

Flanders, H. 1963. *Differential forms*. New York: Academic Press.

Gauss, C. 1811. *Werke* 8, 90-91; a partial translation is in [Birkhoff 1973, 31].

——— 1813. Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata. *Comm. soc. reg. scient. Gottingensis* 2; *Werke* 5, 1-22.

Green, G. 1828. *An Essay on the application of mathematical analysis to the theories of electricity and Magnetism*.

London; *Mathematical Papers*, 3-115.

Helmholtz, H. von. 1858. Über Integrale des hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen. *Crelle* 55, 25-55; a translation by P. Tait is in *Philosophical Magazine* (4) 33 (1868), 485-511.

Jacobi, C. 1836. Über die Reduction der Integration der partiellen Defferentialgleichungen erster Ordnung.... *Crelle* 17, 97-162; *Werke* 4, 57-127.

Katz, V. 1979. The history of Stokes' theorem. *Mathematics Magazine* 52, 146-156.

Lagrange, J. 1760. Application de la méthode exposée dans le mémoire précédent a la solution de différents problèmes de dynamique. *Miscellenea Taurinensia* 2, 196-298.

——— 1788. *Mécanique analytique*, 1st ed. Paris.

——— 1811. *Mécanique analytique*, 2nd ed. Paris: Desaint; *Oeuvres*, 11.

Laplace, P.-S. de. 1799. *Traite de mécanique céleste.* Paris: Crapelet; *Oeuvres*, 2.

Laurent, H. 1888. *Traite d'analyse*, Vol. 3. Paris.

Loria, G. 1915. Commemorazione del compianto Socio prof. Placido Tardy. *Rendiconti Accademia dei Lincei.* (5) 24, 505-521.

Maxwell, C. 1855. On Faraday's lines of force. *Transactions of the Cambridge Philosophical Society* 10; *Scientific Papers* 1, 155-229.

——— 1873. *A treatise on electricity and magnetism.* London: Oxford Univ. Press, (Clarendon).

Ostrogradskii, M. 1826. Démonstration d'un théorème du calcul intégral; published in Russian in [Yushkevich 1965, 49-96].

——— 1836. Mémoire sur le calcul des variations des intégrales multiples. *Mémoires de L'Académie Impériale des Sciences de St. Petersbourg* (6) 3 (Part 1), 36-58; also in *Crelle* 15, 332-354.

——— 1838. Sur la transformation des variables dans les intégrales multiples. *Mémoires de L'Académie Impériale des Sciences de St. Petersbourg, Classé de Physique et de Mathêmatique* 1, 401-407.

——— 1840. Sur une note relative aux intégrales définies déduites de la théorie des surfaces orthogonales. *Bulletin Scientifique Publie par l'Académie des Sciences de St. Petersbourg* 7, No. 24, 362-365.

Poincaré, H. 1886. Sur les résidus des intégrales doubles. *Comptes Rendus* 102, 202-204; *Oeuvres* 3, 437-439.

——— 1887. Sur les résidus des intégrales doubles. *Acta Mathematica* 9, 321-380; *Oeuvres* 3, 440-489.

——— 1895. Analysis situs. *Journal de l'École Polytechnique* (2) 1, 1-121; *Oeuvres* 6, 193-288.

——— 1899. Complément a l'analysis situs. *Rendiconti del Circolo Matematico di Palermo* 13, 285-343; *Oeuvres* 6, 290-337.

——— 1921. Analyse de ses travaux sur les intégrales faite par H. Poincaré. *Acta Mathematica* 38, 73-77; *Oeuvres* 3, 325-329.

Poisson, S. 1829. Mémoire sur l'équilibre et le mouvement des corps élastiques. *Mémoires de l'Académie Royale des Sciences de l'Institut de France* 8, 357-571.

Pont, J.-C. 1974. *La topologie algébrique des origines a Poincaré.* Paris: Presses Universitaires de France.

Riemann, B. 1851. *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* Göttingen; *Werke*, 3-48; a partial translation is given in [Birkhoff 1973, 48-50].

——— 1857. Lehrsätze aus der analysis situs für die Theorie der Integrale von zweigliedrigen vollständigen Differentialen *Crelle* 54, 105-110; *Werke* 91-96; a translation is given in [Birkhoff 1973, 52-56].

Stokes, G. 1845. On the theories of the internal friction of fluids in motion.... *Transactions of the Cambridge Philosophical Society* 8, 287-341; *Mathematics and Physics Papers* 1, 75-129.

——— 1849. On the dynamical theory of diffraction. *Transactions of the Cambridge Philosophical Society* 9, 1-85; *Mathematics and Physics Papers* 2, 243-328.

Stolze, C. 1978. A history of the divergence theorem. *Historia Mathematica* 5, 437-442.

Thomson, W. 1851. A mathematical theory of magnetism. *Philosophical Transactions of the Royal Society of London* 141, 243-285; *Reprints of Papers on Electrostatics and Magnetism*, 340-405.

——— 1869. On vortex motion. *Transactions of the Royal Society of Edinburgh* 25, 217-260; *Mathematics and Physics Papers* 4, 13-66.

——— & Tait, P. 1867. *Treatise on natural philosophy.* Oxford.

Tonelli, A. 1873. Osservazioni sulla teoria della connessione. *Atti Della Reale Accademia dei Lincei* 2, 594-602; a partial translation is in [Birkhoff 1973, 53].

Volterra, V. 1889a. Sur une généralisation de la théorie des fonctions d'une variable imaginaire. *Acta Mathematica* 12, 233-258; *Opere* 1, 377-402.

——— 1889b. Delle variabili complesse negli iperspazi. *Rendiconti Accademia dei Lincei* (4) 5, 158-165; *Opere* 1, 403-410.

——— 1889c. Sulle funzioni coniugate. *Rendiconti Accademia dei Lincei* (4) 5, 599-611; *Opere* 1, 420-432.

Weil, A. 1979. Riemann, Betti and the birth of topology. *Archive for History of Exact Sciences* 20, 91-96.

Yushkevich, A. P. 1965. O neopublikovannykh rannikh rabotakh M. V. Ostrogradskovo. *Istoriko-Matematicheskie Issledovaniya* 16, 11-48.